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Random Point Fields for Para-Particles of order 3

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概 要

Random point fields which describe gases consist of para-particles of order three are given by means of the canonical ensemble approach. The analysis for the case of the para-fermion gases is discussed in full detail.

1 Introduction

The purpose of this note is to apply the method which we have developed in [TIa] to statistical mechanics of gases which consist of para-particles of order 3. We begin with quantum mechanical thermal systems of finite fixed numbers of para-bosons and/or para-fermions in the bounded boxes in $\mathbb{R}^d$. Taking the thermodynamic limits, random point fields on $\mathbb{R}^d$ are obtained. We will see that the point fields obtained in this way are those of $\alpha = \pm 1/3$ given in [ShTa03].

We use the representation theory of the symmetric group. (cf. e.g. [JK81, S91, Si96]) Its basic facts are reviewed briefly, in section 2, along the line on which the quantum theory of para-particles are formulated. We state the results in section 3. Section 4 devoted to the full detail of the discussion on the thermodynamic limits for para-fermion's case.

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2 Brief review on Representation of the symmetric group

We say that $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{N}^n$ is a Young frame of length n for the symmetric group $\mathcal{S}_N$ if

$$\sum_{j=1}^{n} \lambda_j = N, \quad \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0.$$

We associate the Young frame $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ with the diagram of λ_1-boxes in the first row, λ_2-boxes in the second row,..., and λ_n-boxes in the n-th row. A Young tableau on a Young frame is a bijection from the numbers $1, 2, \cdots, N$ to the N boxes of the frame.

Let M_p^N be the set of all the Young frames for $\mathcal{S}_N$ which have lengths less than or equal to p . For each frame in M_p^N, let us choose one tableau from those on the frame. The choices are arbitrary but fixed. $\mathcal{T}_p^N$ denotes the set of all the tableaux chosen in this way. The row stabilizer of tableau T is denoted by $\mathcal{R}(T)$, i.e., the subgroup of $\mathcal{S}_N$ consists of those elements that keep all rows of T invariant, and $\mathcal{C}(T)$ the column stabilizer whose elements preserve all columns of T.

Let us introduce the three elements

$$a(T) = \frac{1}{\#\mathcal{R}(T)} \sum_{\sigma \in \mathcal{R}(T)} \sigma, \qquad b(T) = \frac{1}{\#\mathcal{C}(T)} \sum_{\sigma \in \mathcal{C}(T)} \mathrm{sgn}(\sigma)\sigma$$

and

$$e(T) = \frac{d_T}{N!} \sum_{\sigma \in \mathcal{R}(T)} \sum_{\tau \in \mathcal{C}(T)} \mathrm{sgn}(\tau)\sigma\tau = c_T a(T) b(T)$$

of the group algebra $\mathbb{C}[\mathcal{S}_N]$ for each $T \in \mathcal{T}_p^N$, where d_T is the dimension of the irreducible representation of $\mathcal{S}_N$ corresponding to T and $c_T = d_T \#\mathcal{R}(T) \#\mathcal{C}(T)/N!$. As is known,

$$a(T_1)\sigma b(T_2) = b(T_2)\sigma a(T_1) = 0 \tag{2.1}$$

hold for any $\sigma \in \mathcal{S}_N$ if $T_2 \multimap T_1$. The relations

$$a(T)^2 = a(T), \quad b(T)^2 = b(T), \quad e(T)^2 = e(T), \quad e(T_1)e(T_2) = 0 \quad (T_1 \neq T_2) \tag{2.2}$$

also hold for $T, T_1, T_2 \in \mathcal{T}_p^N$. For later use, let us introduce

$$d(T) = e(T)a(T) = c_T a(T) b(T) a(T) \tag{2.3}$$

for $T \in \mathcal{T}_p^N$. They satisfy

$$d(T)^2 = d(T), \quad d(T_1)d(T_2) = 0 \quad (T_1 \neq T_2) \tag{2.4}$$

which are shown readily from (2.2) and (2.1). The inner product $< \cdot, \cdot >$ of $\mathbb{C}[\mathcal{S}_N]$ is defined by

$$< \sigma, \tau >= \delta_{\sigma\tau} \quad \text{for } \sigma, \tau \in \mathcal{S}_N$$

and the sesqui-linearity.

The left representation L and the right representation R of $\mathcal{S}_N$ on $\mathbb{C}[\mathcal{S}_N]$ are defined by

$$L(\sigma)g = L(\sigma) \sum_{\tau \in \mathcal{S}_N} g(\tau)\tau = \sum_{\tau \in \mathcal{S}_N} g(\tau)\sigma\tau = \sum_{\tau \in \mathcal{S}_N} g(\sigma^{-1}\tau)\tau$$

and

$$R(\sigma)g = R(\sigma) \sum_{\tau \in \mathcal{S}_N} g(\tau)\tau = \sum_{\tau \in \mathcal{S}_N} g(\tau)\tau\sigma^{-1} = \sum_{\tau \in \mathcal{S}_N} g(\tau\sigma)\tau,$$

respectively. Here and hereafter we identify $g : \mathcal{S}_N \to \mathbb{C}$ and $\sum_{\tau \in \mathcal{S}_N} g(\tau)\tau \in \mathbb{C}[\mathcal{S}_N]$. They are extended to the representation of $\mathbb{C}[\mathcal{S}_N]$ on $\mathbb{C}[\mathcal{S}_N]$ as

$$L(f)g = fg = \sum_{\sigma,\tau} f(\sigma)g(\tau)\sigma\tau = \sum_{\sigma} \big(\sum_{\tau} f(\sigma\tau^{-1})g(\tau)\big)\sigma$$

and

$$R(f)g = g\hat{f} = \sum_{\sigma,\tau} g(\sigma)f(\tau)\sigma\tau^{-1} = \sum_{\sigma} \big(\sum_{\tau} g(\sigma\tau)f(\tau)\big)\sigma,$$

where $\hat{f} = \sum_\tau \hat{f}(\tau)\tau = \sum_\tau f(\tau^{-1})\tau = \sum_\tau f(\tau)\tau^{-1}$.

The character of the irreducible representation of $\mathcal{S}_N$ corresponding to tableau $T \in \mathcal{T}_p^N$ is obtained by

$$\chi_T(\sigma) = \sum_{\tau \in \mathcal{S}_N} (\tau, L(\sigma)R(e(T))\tau) = \sum_{\tau \in \mathcal{S}_N} (\tau, \sigma\tau\widehat{e(T)}).$$

We introduce a tentative notation

$$\chi_g(\sigma) \equiv \sum_{\tau \in \mathcal{S}_N} (\tau, L(\sigma)R(g)\tau) = \sum_{\tau,\gamma \in \mathcal{S}_N} (\tau, \sigma\tau\gamma^{-1})g(\gamma) = \sum_{\tau \in \mathcal{S}_N} g(\tau^{-1}\sigma\tau) \qquad (2.5)$$

for $g = \sum_\tau g(\tau)\tau \in \mathbb{C}[\mathcal{S}_N]$. Then $\chi_T = \chi_{e(T)}$ holds.

Now let us consider representations of $\mathcal{S}_N$ on Hilbert spaces. Let $\mathcal{H}_L$ be a certain L^2 space which will be specified in the next section and $\otimes^N \mathcal{H}_L$ its N-fold Hilbert space tensor product. Let U be the representation of $\mathcal{S}_N$ on $\otimes^N \mathcal{H}_L$ defined by

$$U(\sigma)\varphi_1 \otimes \cdots \otimes \varphi_N = \varphi_{\sigma^{-1}(1)} \otimes \cdots \otimes \varphi_{\sigma^{-1}(N)} \qquad \text{for } \varphi_1, \cdots, \varphi_N \in \mathcal{H}_L,$$

or equivalently by

$$(U(\sigma)f)(x_1, \cdots, x_N) = f(x_{\sigma(1)}, \cdots, x_{\sigma(N)}) \qquad \text{for } f \in \otimes^N \mathcal{H}_L.$$

Obviously, U is unitary: $U(\sigma)^* = U(\sigma^{-1}) = U(\sigma)^{-1}$. We extend U for $\mathbb{C}[\mathcal{S}_N]$ by linearity. Then $U(a(T))$ is an orthogonal projection because of $U(a(T))^* = U(\widehat{a(T)}) = U(a(T))$ and (2.2). So are $U(b(T))$'s, $U(d(T))$'s and $P_{pB} = \sum_{T \in \mathcal{T}_p^N} U(d(T))$. Note that $\operatorname{Ran} U(d(T)) = \operatorname{Ran} U(e(T))$ because of $d(T)e(T) = e(T)$ and $e(T)d(T) = d(T)$.

3 Para-statistics and Random point fields

3.1 Para-bosons of order 3

Let us consider a quantum system of N para-bosons of order p in the box $\Lambda_L = [-L/2, L/2]^d \subset \mathbb{R}^d$. We refer the literatures [MeG64, HaT69, StT70] for quantum mechanics of para-particles. (See also [OK69].) The arguments of these literatures indicate that the state space of our system is given by $\mathcal{H}_{L,N}^{pB} = P_{pB} \otimes^N \mathcal{H}_L$, where $\mathcal{H}_L = L^2(\Lambda_L)$ with Lebesgue measure is the state space of one particle system in Λ_L. We need the heat operator $G_L = e^{\beta\Delta_L}$ in Λ_L, where Δ_L is the Laplacian in Λ_L with periodic boundary conditions.

It is obvious that there is a CONS of $\mathcal{H}_{L,N}^{pB}$ which consists of the vectors of the form $U(d(T))\varphi_{k_1}^{(L)} \otimes \cdots \otimes \varphi_{k_N}^{(L)}$, which are the eigenfunctions of $\otimes^N G_L$. Then, we define the point field $\mu_{L,N}^{pB}$ of N free para-bosons of order p as in section 2 of [TIa] and its generating functional is given by

$$\int e^{-<f,\xi>} d\mu_{L,N}^{pB}(\xi) = \frac{\mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N \tilde{G}_L) P_{pB}]}{\mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) P_{pB}]},$$

where f is a nonnegative continuous function on Λ_L and $\tilde{G}_L = G_L^{1/2} e^{-f} G_L^{1/2}$.

Lemma 3.1

$$\int e^{-<f,\xi>} d\mu_{L,N}^{pB}(\xi) = \frac{\sum_{T\in\mathcal{T}_p^N} \sum_{\sigma\in\mathcal{S}_N} \chi_T(\sigma) \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N \tilde{G}_L) U(\sigma)]}{\sum_{T\in\mathcal{T}_p^N} \sum_{\sigma\in\mathcal{S}_N} \chi_T(\sigma) \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(\sigma)]} \tag{3.1}$$

$$= \frac{\sum_{T\in\mathcal{T}_p^N} \int_{\Lambda_L^N} \det{}_T\{\tilde{G}_L(x_i, x_j)\}_{1\leqslant i,j\leqslant N} dx_1 \cdots dx_N}{\sum_{T\in\mathcal{T}_p^N} \int_{\Lambda_L^N} \det{}_T\{G_L(x_i, x_j)\}_{1\leqslant i,j\leqslant N} dx_1 \cdots dx_N} \tag{3.2}$$

Remark 1 : $\mathcal{H}_{L,N}^{pB} = P_{pB} \otimes^N \mathcal{H}_L$ is determined by the choice of the tableaux T's. The spaces corresponding to different choices of tableaux are different subspaces of $\otimes^N \mathcal{H}_L$. However, they are unitarily equivalent and the generating functional given above is not affected by the choice. In fact, $\chi_T(\sigma)$ depends only on the frame on which the tableau T is defined.

Remark 2 : $\det_T A = \sum_{\sigma\in\mathcal{S}_N} \chi_T(\sigma) \prod_{i=1}^N A_{i\sigma(i)}$ in (3.2) is called immanant.

Proof : Since $\otimes^N G$ commutes with $U(\sigma)$ and $a(T)e(T) = e(T)$, we have

$$\mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(d(T))] = \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(e(T)) U(a(T))]$$

$$= \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[U(a(T))(\otimes^N G_L) U(e(T))] = \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(e(T))]. \tag{3.3}$$

On the other hand, we get from (2.5) that

$$\sum_{\sigma\in S_N} \chi_g(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G_L)U(\sigma)] = \sum_{\tau,\sigma\in S_N} g(\tau^{-1}\sigma\tau)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G_L)U(\sigma)]$$
$$= \sum_{\tau,\sigma} g(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G_L)U(\tau\sigma\tau^{-1})] = \sum_{\tau,\sigma} g(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G_L)U(\tau)U(\sigma)U(\tau^{-1})]$$
$$= N!\sum_{\sigma} g(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G_L)U(\sigma)] = N!\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G_L)U(g)], \tag{3.4}$$

where we have used the cyclicity of the trace and the commutativity of $U(\tau)$ with $\otimes^N G$. Putting $g = e(T)$ and using (3.3) and $P_{pB} = \sum_{T\in\mathcal{T}_p^N} U(d(T))$, we obtain the first equation. The second one is obvious. □

Now, let us consider the thermodynamic limit

$$L,\ N \to \infty, \quad N/L^d \to \rho > 0. \tag{3.5}$$

We need the heat operator $G = e^{\beta\Delta}$ on $L^2(\mathbb{R}^d)$. In the following, f is a nonnegative continuous function having a compact support. It is supposed to be fixed in the thermodynamic limit. Its support will be contained in Λ_L for large enough L.

We get the limiting random point field μ_ρ^{3B} on $\mathbb{R}^d$ for the low density region.

Theorem 3.2 *The finite random point field for para-bosons of order* 3 *defined above converge weakly to the random point field whose Laplace transform is given by*

$$\int e^{-<f,\xi>} d\mu_\rho^{3B}(\xi) = \mathrm{Det}\big[1 + \sqrt{1-e^{-f}}\, r_* G(1-r_*G)^{-1}\sqrt{1-e^{-f}}\big]^{-3}$$

in the thermodynamic limit, where $r_* \in (0,1)$ *is determined by*

$$\frac{\rho}{3} = \int \frac{dp}{(2\pi)^d}\frac{r_* e^{-\beta|p|^2}}{1-r_* e^{-\beta|p|^2}} = (r_*G(1-r_*G)^{-1})(x,x),$$

if

$$\frac{\rho}{3} < \rho_c = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d}\frac{e^{-\beta|p|^2}}{1-e^{-\beta|p|^2}}.$$

Remark : The high density region $\rho \geqslant 3\rho_c$ is related to the Bose-Einstein condensation. We need a different analysis for the region. See [TIb] for the case of $p = 1$ and 2.

3.2 Para-fermions of order 3

For Young tableau T, T' denotes the tableau obtained by exchanging the rows and the columns of T, i.e., T' is the transpose of T. The transpose λ' of the frame λ can be defined similarly. Then, T' lives on λ' if T lives on λ. It is obvious that

$$\mathcal{R}(T') = \mathcal{C}(T), \qquad \mathcal{C}(T') = \mathcal{R}(T).$$

The generating functional of the point field $\mu^{pF}_{L,N}$ for N para-fermions of order p in the box Λ_L is given by

$$\int e^{-<f,\xi>} d\mu^{pF}_{L,N}(\xi) = \frac{\sum_{T\in\mathcal{T}_p^N} \mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N \tilde{G})U(d(T'))]}{\sum_{T\in\mathcal{T}_p^N} \mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G)U(d(T'))]}$$

as in the case of para-bosons of order p. And the following expressions also hold.

Lemma 3.3

$$\int e^{-<f,\xi>} d\mu^{pF}_{L,N}(\xi) = \frac{\sum_{T\in\mathcal{T}_p^N}\sum_{\sigma\in S_N}\chi_{T'}(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N \tilde{G}_L)U(\sigma)]}{\sum_{T\in\mathcal{T}_p^N}\sum_{\sigma\in S_N}\chi_{T'}(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G_L)U(\sigma)]} \tag{3.6}$$

$$= \frac{\sum_{T\in\mathcal{T}_p^N}\int_{\Lambda_L^N} \det_{T'}\{\tilde{G}_L(x_i,x_j)\}dx_1\cdots dx_N}{\sum_{T\in\mathcal{T}_p^N}\int_{\Lambda_L^N} \det_{T'}\{G_L(x_i,x_j)\}dx_1\cdots dx_N} \tag{3.7}$$

Theorem 3.4 *The finite random point fields for para-fermions of order* 3 *defined above converge weakly to the point field* μ^{3F}_ρ *whose Laplace transform is given by*

$$\int e^{-<f,\xi>} d\mu^{3F}_\rho(\xi) = \mathrm{Det}\big[1-\sqrt{1-e^{-f}}r_*G(1+r_*G)^{-1}\sqrt{1-e^{-f}}\big]^3$$

in the thermodynamic limit (3.5), where $r_*\in(0,\infty)$ *is determined by*

$$\frac{\rho}{3} = \int\frac{dp}{(2\pi)^d}\frac{r_*e^{-\beta|p|^2}}{1+r_*e^{-\beta|p|^2}} = (r_*G(1+r_*G)^{-1})(x,x). \tag{3.8}$$

4 Proof of Theorem 3.4

In the rest of this paper, we use results in [TIa] frequently. We refer them e.g., Lemma I.3.2 for Lemma 3.2 of [TIa]. Let ψ_T be the character of the induced representation $\mathrm{Ind}^{S_N}_{\mathcal{R}(T)}[\mathbf{1}]$, where $\mathbf{1}$ is the one dimensional representation $\mathcal{R}(T)\ni\sigma\to 1$, i.e.,

$$\psi_T(\sigma) = \sum_{\tau\in S_N} <\tau, L(\sigma)R(a(T))\tau> = \chi_{a(T)}(\sigma).$$

Since the characters χ_T and ψ_T depend only on the frame on which the tableau T lives, not on T itself, we use the notation χ_λ and ψ_λ ($\lambda\in M^N_p$) instead of χ_T and ψ_T, respectively.

Let δ be the frame $(p-1,\cdots,2,1,0)\in M_p^N$. Generalize ψ_μ to those $\mu=(\mu_1,\cdots,\mu_p)\in \mathbb{Z}^p$ which satisfies $\sum_{j=1}^p \mu_j = N$ by

$$\psi_\mu = 0 \qquad \text{for } \mu\in\mathbb{Z}^p-\mathbb{Z}_+^p$$

and

$$\psi_\mu=\psi_{\pi\mu} \quad \text{for} \quad \mu\in\mathbb{Z}_+^p \quad \text{and} \quad \pi\in\mathcal{S}_p \quad \text{such that} \quad \pi\mu\in M_p^N,$$

where $\mathbb{Z}_+=\{0\}\cup\mathbb{N}$. Then the determinantal form [JK81] can be written as

$$\chi_\lambda=\sum_{\pi\in\mathcal{S}_p}\operatorname{sgn}\pi\,\psi_{\lambda+\delta-\pi\delta}. \tag{4.1}$$

Let us recall the relations

$$\chi_{T'}(\sigma)=\operatorname{sgn}\sigma\,\chi_T(\sigma), \qquad \varphi_{T'}(\sigma)=\operatorname{sgn}\sigma\,\psi_T(\sigma),$$

where

$$\varphi_{T'}(\sigma)=\sum_\tau <\tau, L(\sigma)R(b(T'))\tau>=\chi_{b(T')}(\sigma)$$

denotes the character of the induced representation $\mathrm{Ind}_{\mathcal{C}(T')}^{\mathcal{S}_N}[\operatorname{sgn}]$, where sgn is the representation $\mathcal{C}(T')=\mathcal{R}(T)\ni\sigma\mapsto\operatorname{sgn}\sigma$. Then we have a variant of (4.1)

$$\chi_{\lambda'}=\sum_{\pi\in\mathcal{S}_p}\operatorname{sgn}\pi\,\varphi_{\lambda'+\delta'-(\pi\delta)'}. \tag{4.2}$$

Now we consider the denominator of (3.6). Let $T\in\mathcal{T}_p^N$ live on $\mu=(\mu_1,\cdots,\mu_p)\in M_p^N$. Thanks to (3.4) for $g=b(T')$, we have

$$\sum_{\sigma\in\mathcal{S}_N}\varphi_{T'}(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}((\otimes^N G)U(\sigma)) = N!\mathrm{Tr}_{\otimes^N\mathcal{H}_L}((\otimes^N G)U(b(T')))$$

$$= N!\prod_{j=1}^p \mathrm{Tr}_{\otimes^{\mu_j}\mathcal{H}_L}((\otimes^{\mu_j}G)A_{\mu_j}),$$

where $A_n=\sum_{\tau\in\mathcal{S}_n}\operatorname{sgn}(\tau)U(\tau)/n!$ is the anti-symmetrization operator on $\otimes^n\mathcal{H}_L$. In the last step, we have used

$$b(T')=\prod_{j=1}^p\sum_{\sigma\in\mathcal{R}_j}\frac{\operatorname{sgn}\sigma}{\#\mathcal{R}_j}\sigma,$$

where $\mathcal{R}_j$ is the symmetric group of μ_j numbers which lie on the j-th row of the tableau T. Then (4.2) yields

$$\sum_{\sigma\in\mathcal{S}_N}\chi_{\lambda'}(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G_L)U(\sigma)]=\sum_{\pi\in\mathcal{S}_p}\operatorname{sgn}\pi\sum_{\sigma\in\mathcal{S}_N}\varphi_{\lambda'+\delta'-(\pi\delta)'}(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G_L)U(\sigma)]$$

$$= N! \sum_{\pi \in S_p} \operatorname{sgn} \pi \prod_{j=1}^{p} \operatorname{Tr}_{\otimes^{\lambda_j - j + \pi(j)} \mathcal{H}_L} \left((\otimes^{\lambda_j - j + \pi(j)} G_L) A_{\lambda_j - j + \pi(j)} \right).$$

Here we understand that $\operatorname{Tr}_{\otimes^n \mathcal{H}_L} \left((\otimes^n G) A_n \right) = 1$ if $n = 0$ and $= 0$ if $n < 0$ in the last expression. Let us recall the defining formula of Fredholm determinant

$$\operatorname{Det}(1 + J) = \sum_{n=0}^{\infty} \operatorname{Tr}_{\otimes^n \mathcal{H}} [(\otimes^n J) A_n]$$

for a trace class operator J. We use it in the form

$$\operatorname{Tr}_{\otimes^n \mathcal{H}} [(\otimes^n G_L) A_n] = \oint_{S_r(0)} \frac{dz}{2\pi i z^{n+1}} \operatorname{Det}(1 + z G_L), \tag{4.3}$$

where $r > 0$ can be set arbitrary. Note that the right hand side equals to 1 for $n = 1$ and to 0 for $n < 0$. Then we have the following expression of the denominator of (3.6)

$$\begin{aligned}
&\sum_{\lambda \in \mathcal{M}_p^N} \sum_{\sigma \in S_N} \chi_{\lambda'}(\sigma) \operatorname{Tr}_{\otimes^N \mathcal{H}_L} [(\otimes^N G_L) U(\sigma)] \\
&= N! \sum_{\lambda \in \mathcal{M}_p^N} \sum_{\pi \in S_p} \operatorname{sgn}\pi \oint \cdots \oint_{S_r(0)^p} \prod_{j=1}^{p} \frac{\operatorname{Det}(1 + z_j G_L)\, dz_j}{2\pi i z_j^{\lambda_j - j + \pi(j) + 1}}. \\
&= N! \sum_{\lambda \in \mathcal{M}_p^N} \oint \cdots \oint_{S_r(0)^p} \frac{[\prod_{1 \leqslant i < j \leqslant p} (z_i - z_j)] [\prod_{j=1}^{p} \operatorname{Det}(1 + z_j G_L)] dz_1 \cdots dz_p}{\prod_{j=1}^{p} 2\pi i z_j^{\lambda_j + p - j + 1}}.
\end{aligned} \tag{4.4}$$

The similar formula for the numerator also holds.

Now we concentrate on the case of $p = 3$. To make the thermodynamic limit procedure explicit, let us take a sequence $\{L_N\}_{N \in \mathbb{N}}$ which satisfies $N / L_N^d \to \rho$ as $N \to \infty$. In the followings, $r = r_k \in [0, \infty)$ denotes the unique solution of

$$\operatorname{Tr} r G_{L_N} (1 + r G_{L_N})^{-1} = k \tag{4.5}$$

for $0 \leqslant k \leqslant N$. We suppress the N dependence of r_k. The existence and the uniqueness of the solution follow from the fact that the left-hand side of (4.5) is a continuous and monotone function of r. See Lemma I.3.2, for details. We put

$$v_k = \operatorname{Tr} [r_k G_{L_N} (1 + r_k G_{L_N})^{-2}] \tag{4.6}$$

and

$$\mathcal{D}_{k,l,m} = \oint \oint \oint_{S_r(0)^3} \frac{[\prod_{j=1}^{3} \operatorname{Det}(1 + z_j G_{L_N})] (z_1 - z_2)(z_2 - z_3)}{(2\pi i)^3 z_1^{k+1} z_2^{l+1} z_3^{m+1}} dz_1 dz_2 dz_3,$$

for $k,l,m\in\mathbb{Z}$. Note that $\mathcal{D}_{k,l,m}=0$ if at least one of k,l,m is negative. Summing over λ_1 and λ_3 in (4.4) for $p=3$, we get

$$\sum_{\lambda\in M_3^N}\sum_{\sigma\in S_N}\chi_{\lambda'}(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_{L_N}}[(\otimes^N G_{L_N})U(\sigma)]=N!\Big(\sum_{l=1}^{[N/3]+1}\mathcal{D}_{N+3-2l,l,l-1}+\sum_{l=[N/3]+2}^{[N/2]+1}\mathcal{D}_{l,l,N+2-2l}\Big).$$

Since $r>0$ of the contour $S_r(0)$ is arbitrary, we may change the complex integral variables $z_j=r_j\eta_j$ with $\eta_j\in S_1(0)$ for $j=1,2,3$. Thanks to the property of Fredholm determinant, we have

$$\mathrm{Det}[1+z_jG_{L_N}]=\mathrm{Det}[1+r_jG_{L_N}]\,\mathrm{Det}[1+(\eta_j-1)r_jG_{L_N}(1+r_jG_{L_N})^{-1}]$$

Now, we can put

$$\mathcal{F}_{k,l,m}=\frac{r_0^{3k_0}v_0^{5/2}}{\mathrm{Det}[1+r_0G_{L_N}]^3}\mathcal{D}_{k,l,m}=R_{k,l,m}v_0^{5/2}\,I_{k,l,m},$$

where

$$R_{k_1,k_2,k_3}=\prod_{j=1}^{3}\frac{r_0^{k_0}\mathrm{Det}[1+r_jG_{L_N}]}{r_j^{k_j}\mathrm{Det}[1+r_0G_{L_N}]}$$

and

$$I_{k_1,k_2,k_3}=\oint\oint\oint_{S_1(0)^3}\Big(\prod_{j=1}^{3}\mathrm{Det}[1+(\eta_j-1)r_jG_{L_N}(1+r_jG_{L_N})^{-1}]\Big)$$
$$\times(r_1\eta_1-r_2\eta_2)(r_2\eta_2-r_3\eta_3)\frac{d\eta_1d\eta_2d\eta_3}{(2\pi i)^3\eta_1^{k_1+1}\eta_2^{k_2+1}\eta_3^{k_3+1}}.$$

Here $k_0=(N+2)/3$ and $k_1,k_2,k_3\in\mathbb{Z}_+$ satisfy $k_1\geqslant k_2\geqslant k_3$ and $k_1+k_2+k_3=3k_0$. We use the abbreviation r_ν and v_ν for r_{k_ν} and $v_{k_\nu}(\nu=0,1,2,3)$, respectively. Here, let us recall that $r_0\to r_*$ in the thermodynamic limit because of $k_0/L^d\to\rho/3$, (3.8) and Lemma I.3.5.

Define a sequence $\{f_N\}_{N\in\mathrm{N}}$ of nonnegative functions on $\mathbb{R}$ by

$$f_N(x)=\begin{cases}\mathcal{F}_{l,l,N+2-2l} & \text{for } \sqrt{N+2}\,x\in[l-1-(N+2)/3,l-(N+2)/3)\\ & \text{and}\quad l=[N/3]+2,\cdots,[N/2]+1\\ \mathcal{F}_{N+3-2l,l,l-1} & \text{for } \sqrt{N+2}\,x\in[l-1-(N+2)/3,l-(N+2)/3)\\ & \text{and}\quad l=1,2,\cdots,[N/3]+1\\ 0 & \text{otherwise.}\end{cases}$$

Then the denominator of (3.6) becomes

$$N!\sqrt{N+2}\,\frac{\mathrm{Det}[1+r_0G_{L_N}]^3}{r_0^{3k_0}v_0^{5/2}}\int_{-\infty}^{\infty}f_N(x)\,dx$$

We introduce $\tilde{\mathcal{D}}_{k,l,m}, \tilde{\mathcal{F}}_{k,l,m}$ and $\tilde{f}_N$ using $\tilde{G}_{L_N}$ instead of G_{L_N} in $\mathcal{D}_{k,l,m}, \mathcal{F}_{k,l,m}$ and f_N and so on, to get the expression

$$\mathrm{E}^{3F}_{L,N}\left[e^{-<f,\xi>}\right] = \frac{\mathrm{Det}[1+\tilde{r}_0\tilde{G}_{L_N}]^3}{\mathrm{Det}[1+r_0 G_{L_N}]^3}\frac{r_0^{3k_0}}{\tilde{r}_0^{3k_0}}\frac{v_0^{5/2}}{\tilde{v}_0^{5/2}}\frac{\int_{-\infty}^{\infty}\tilde{f}_N(x)\,dx}{\int_{-\infty}^{\infty}f_N(x)\,dx}.$$

From Lemma I.3.6, we have

$$\frac{\tilde{v}_0}{v_0} \to 1 \tag{4.7}$$

in the thermodynamic limit. Similarly, we obtain

$$\frac{r_0^{k_0}\,\mathrm{Det}[1+\tilde{r}_0\tilde{G}_{L_N}]}{\tilde{r}_0^{k_0}\,\mathrm{Det}[1+r_0 G_{L_N}]} \to \mathrm{Det}\left[1-\sqrt{1-e^{-f}}r_* G(1+r_*G)^{-1}\sqrt{1-e^{-f}}\right]$$

from the proof of Theorem I.3.1 (see Eq. (a–c), where we should read N as k_0, z_N as r_0 and $\alpha=-1$). Thus Theorem 3.4 is proved, if we get the following lemma:

Lemma 4.1 *Under the thermodynamic limit,*

$$\int_{-\infty}^{\infty}\tilde{f}_N(x)\,dx,\ \int_{-\infty}^{\infty}f_N(x)\,dx \to \int_{-\infty}^{\infty}e^{-2\rho x^2/\kappa}\frac{dx}{(2\pi)^{3/2}}$$

hold, where

$$\kappa = \int\frac{dp}{(2\pi)^d}\frac{r_* e^{-\beta|p|^2}}{(1+r_*e^{-\beta|p|^2})^2}.$$

Proof: Let $k, r, v \in [0,\infty)$ satisfy the relations

$$k = \mathrm{Tr}\,[rG_{L_N}(1+rG_{L_N})^{-1}], \quad v = \mathrm{Tr}\,[rG_{L_N}(1+rG_{L_N})^{-2}]. \tag{4.8}$$

1° *There exist positive constants c_1 and c_2 which depend only on the density ρ such that*

$$r_j \leqslant c_1, \quad r_j - r_l \leqslant c_1\frac{k_j-k_l}{k_l}, \quad c_2 k_j \leqslant v_j \leqslant k_j,$$

hold for $k_j, k_l > 0$ satisfying $k_j > k_l$.

We have $v \leqslant k$ and $r \leqslant r_N$ for $k \leqslant N$. Recall r_N converges to the constant r^* which determined by

$$\int\frac{dp}{(2\pi)^d}\frac{r^*e^{-\beta|p|^2}}{1+r^*e^{-\beta|p|^2}} = \rho.$$

Then $\{r_N\}$ is bounded from above. Hence we have $r \leqslant r_N \leqslant c_1$ and $v \geqslant k/(1+r_N) \geqslant k/(1+c_1)$ since $0 \leqslant G_{L_N} \leqslant 1$. Thanks to $dk/dr = v/r \geqslant k/c_1$, we get $c_1\int_{k_l}^{k_j}dk/k \geqslant \int_{r_k}^{r_j}dr$, which yields the second inequality. ◊

2° *There exist positive constants c'_0, c'_1 and c'_2 which depend only on ρ such that*

$$A_{k,n} = \oint_{S_1(0)} \mathrm{Det}[1+(\eta-1)rG_{L_N}(1+rG_{L_N})^{-1}]\frac{(\eta-1)^n\,d\eta}{2\pi i\eta^{k+1}} \quad (n=0,1,2,\ k=0,1,\cdots,N)$$

satisfy

$$A_{k,0} = (1+o(1))/\sqrt{2\pi v}, \quad A_{k,2} = (-1+o(1))/\sqrt{2\pi v^3} \quad \textit{for large } k \leqslant N$$

and

$$|A_{k,0}| \leqslant c'_0/\sqrt{1+k}, \quad |A_{k,1}| \leqslant c'_1/\sqrt{1+k}^3,$$
$$|A_{k,2}| \leqslant c'_2/\sqrt{1+k}^3 \quad \textit{for all } k=0,1,\cdots,N.$$

Put

$$h_k(x) = \chi_{[-\pi\sqrt{v},\pi\sqrt{v}]}(x)e^{-ikx/\sqrt{v}}\mathrm{Det}[1+(e^{ix/\sqrt{v}}-1)rG_{L_N}(1+rG_{L_N})^{-1}],$$

as in the proof of Proposition I.A.2. Then, we have

$$|h_k(x)| \leqslant e^{-2x^2/\pi^2} \in L^1(\mathbb{R}) \tag{4.9}$$

and

$$h_k(x) = \chi_{[-\pi\sqrt{v},\pi\sqrt{v}]}(x)e^{-x^2/2}e^{\delta} \to e^{-x^2/2} \quad \text{as} \quad N \geqslant k \to \infty \tag{4.10}$$

where $|\delta| \leqslant 4|x^3|/9\sqrt{3v}$.

Setting $\eta = \exp(ix/\sqrt{v})$, we have

$$A_{k,n} = \int_{\infty}^{\infty} \frac{(e^{ix/\sqrt{v}}-1)^n h_k(x)}{2\pi\sqrt{v}}\,dx.$$

Then, $|A_{k,0}| \leqslant c'/\sqrt{v} \leqslant c''/\sqrt{k}$ for $k=1,2,\cdots,N$. On the other hand, Cauchy's integral formula yields $A_{0,0}=1$, readily. So we get the bound $|A_{k,0}| \leqslant c'_0/\sqrt{1+k}$.

Now the asymptotic behavior of $A_{k,0}$ can be derived by the use of dominated convergence theorem and (4.10).

For $n=1$, we have

$$A_{k,1} = \frac{i}{2\pi v}\int_{-\infty}^{\infty} xh_k(x)\,dx + R,$$

where

$$|R| \leqslant \int \frac{x^2}{4\pi\sqrt{v^3}}h_k(x)\,dx = O(1/\sqrt{v^3}).$$

The integrand of first term can be written as

$$xh_k(x) = \chi_{[-\pi\sqrt{v}/3,\pi\sqrt{v}/3]}(x)xe^{-x^2/2} + \chi_{[-\pi\sqrt{v}/3,\pi\sqrt{v}/3]}(x)x(e^{\delta}-1)e^{-x^2/2}$$

$$+\chi_{[-\pi\sqrt{v},-\pi\sqrt{v}/3]\cup[\pi\sqrt{v}/3,\pi\sqrt{v}]}(x)\pi\sqrt{v}h_k(x).$$

The integral of the first term of the right hand side is 0. While the second term is bounded by $|x\delta|h(x)$, since $|e^\delta - 1| \leqslant |\delta|e^{\delta \vee 0}$. For the third term, we use (4.9). Then we get the bound $|\int xh_k(x)\,dx| \leqslant c'''/\sqrt{v}$ for $k \geqslant 1$. Together with $A_{0,1} = 0$, the bounds for $A_{k,1}$ are derived. Similarly, we get the formulae for $A_{k,2}$. $\Diamond$

3° *Let $(k_1, k_2, k_3) \in \mathbb{Z}_+$ satisfies*

$$k_1 \geqslant k_2 \geqslant k_3, \quad k_1 + k_2 + k_3 = 3k_0 = N + 2$$

and

$$k_1 = k_2 \text{ or } k_2 = k_3 + 1.$$

Then the estimates

$$|v_0^{5/2} I_{k_1,k_2,k_3}| \leqslant c\left(\frac{k_0}{1+k_3}\right)^{5/2} \leqslant c' e^{(k_0-k_3)^2/4k_0}$$

hold for all such (k_1, k_2, k_3) and

$$v_0^{5/2} I_{k_1,k_2,k_3} = \frac{v_0^{5/2}(1+o(1))}{(2\pi)^{3/2} v_1^{1/2} v_2^{3/2} v_3^{1/2}}$$

holds for large N and (k_1, k_2, k_3), where c, c' are positive constants depending only on ρ.

In fact, expanding

$$(r_1\eta_1 - r_2\eta_2)(r_2\eta_2 - r_3\eta_3) = (r_1(\eta_1 - 1) - r_2(\eta_2 - 1) + r_1 - r_2)(r_2(\eta_2 - 1) - r_3(\eta_3 - 1) + r_2 - r_3)$$

in the integrand of I_{k_1,k_2,k_3}, we get the first inequality from 1° and 2°. The second inequality is obvious. Similarly, the asymptotic behavior follows. $\Diamond$

4°

$$R_{k_1,k_2,k_3} = e^{-\sum_{j=1}^3 (k_0-k_j)^2/2v_j'}$$

holds where $v_j' = \mathrm{Tr}\,[r_j' G_{L_N}(1 + r_j' G_{L_N})^{-2}]$ for a certain middle point r_j' between r_0 and r_j. Especially, we have the bound

$$R_{k_1,k_2,k_3} \leqslant e^{-(k_0-k_3)^2/2k_0}.$$

Recall that G_{L_N} is a non-negative trace class self-adjoint operator. If we put

$$\psi(t) = \log \mathrm{Det}[1 + e^t G_{L_N}] = \mathrm{Tr}\,[\log(1 + e^t G_{L_N})],$$

we have

$$\psi'(t) = \mathrm{Tr}\,[e^t G_{L_N}(1 + e^t G_{L_N})^{-1}], \quad \psi''(t) = \mathrm{Tr}\,[e^t G_{L_N}(1 + e^t G_{L_N})^{-2}].$$

In the equality

$$\psi(t) - t\psi'(t) - \psi(t_0) + t_0\psi'(t_0) = \int_t^{t_0} (s - t_0)\psi''(s)\, ds + t_0(\psi'(t_0) - \psi'(t)),$$

apply

$$\int_t^{t_0} (s - t_0)\psi''(s)\, ds = \int_t^{t_0} ds \int_{t_0}^{s} du\psi''(s)\frac{\psi''(u)}{\psi''(u)} = -\frac{(\psi'(t) - \psi'(t_0))^2}{2\psi''(u_c)},$$

where u_c is a middle point of t and t_0. Then we obtain

$$\begin{aligned}\frac{e^{t_0\psi'(t_0)}}{e^{t\psi'(t)}}\frac{\mathrm{Det}[1 + e^t G_{L_N}]}{\mathrm{Det}[1 + e^{t_0} G_{L_N}]} &= e^{\psi(t) - t\psi'(t) - \psi(t_0) + t_0\psi'(t_0)} \\ &= e^{t_0(\psi'(t_0) - \psi'(t)) - (\psi'(t) - \psi'(t_0))^2/2\psi''(u_c)}.\end{aligned}$$

Set $e^t = r_j$ and $e^{t_0} = r_0$. Then $\psi'(t) = k_j, \psi'(t_0) = k_0, \psi''(t) = v_j$ and $\psi''(t_0) = v_0$ hold. Taking the product of those equalities for $j = 1, 2$ and 3, we get the desired expression, since $3k_0 = k_1 + k_2 + k_3$. $\Diamond$

5° Recall that the functions $\varphi_k^{(L)}(x) = L^{-d/2} \exp(i2\pi k \cdot x/L) \quad (k \in \mathbb{Z}^d)$ constitute a C.O.N.S. of $L^2(\Lambda_L)$, where $G_L\varphi_k^{(L)} = e^{-\beta|2\pi k/L|^2}\varphi_k^{(L)}$ holds for all $k \in \mathbb{Z}^d$. Then, we obtain

$$\frac{v_0}{L^d} = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \left(\frac{2\pi}{L}\right)^d \frac{r_0 e^{-\beta|2\pi k/L|^2}}{1 + r_0 e^{-\beta|2\pi k/L|^2}} \to \kappa,$$

in the thermodynamic limit, since $k_0/L^d \to \rho/3$ and $r_0 \to r_*$ hold.

From 3° and 4°, we have a bound

$$|F_{k_1,k_2,k_3}| \leqslant c' e^{-(k_0 - k_3)^2/4k_0} \tag{4.11}$$

and

$$F_{k_1,k_2,k_3} = \frac{v_0^{5/2}(1 + o(1))}{(2\pi)^{3/2} v_1^{1/2} v_2^{3/2} v_3^{1/2}} e^{-\sum_j (k_0 - k_j)/2v_j'} \tag{4.12}$$

for large N, k_1, k_2, k_3, where v_j' is a mean value which we have written $\psi''(u_c)$ in 4°. For $l = 1, 2, \cdots, [N/3] + 1$, $\sqrt{N+2}x \in [l - 1 - (N+2)/3, l - (N+2)/3)$ implies $|l - 1 - (N+2)/3| \geqslant \sqrt{N+2}|x|$, hence we get the bound

$$f_N(x) = F_{N+3-2l,l,l-1} \leqslant c' e^{-(N+2)x^2/4k_0} \leqslant c' e^{-3x^2/4}.$$

We also get $f_N(x) \leqslant c' \exp(-3x^2/4)$ for the other cases, similarly.

For fixed $x \in \mathbb{R}$, we choose $l \in \mathbb{Z}$ such that $\sqrt{N+2}x \in [l-1-(N+2)/3, l-(N+2)/3)$. Then we have $v_j/v_0 \to 1$ $(j = 1, 2, 3)$ and

$$\sum_{j=1}^{3} \frac{(k_0 - k_j)^2}{v_j'} = \frac{4N}{v_0}x^2 + o(1).$$

Hence, we obtain $f_N(x) \to (2\pi)^{-3/2}\exp(-2\rho x^2/\kappa)$ in the thermodynamic limit. Thus the dominated convergence theorem yields the desired result for f_N. Because of (4.7), the one for $\tilde{f}_N$ can be proved similarly. $\square$

参考文献

[HaT69] J.B. Hartle and J.R. Taylor, Quantum mechanics of paraparticles, Phys. Rev. **178** (1969) 2043–2051.

[JK81] G. James and A. Kerber, *The Representation Theory of the Symmetric Group (Encyclopedia of mathematics and its applications vol. 16)* (Addison-Wesley, London, 1981)

[MeG64] A.M.L. Messiah and O.W. Greenberg, Symmetrization postulate and its experimental foundation, Phys. Rev. **136** (1964) B248–B267.

[OK69] Y. Ohnuki and S. Kamefuchi, Wavefunctions of identical particles, Ann. Phys. **51** (1969) 337–358.

[S91] B.E. Sagan, *The Symmetric Group* (Brooks/Cole, Pacific Grove, CA, 1991).

[ShTa03] T. Shirai and Y. Takahashi, Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes, J. Funct. Anal. **205** (2003) 414–463.

[Si96] B. Simon, *Representations of Finite and Compact Groups* (A. M. S., Providence, 1996).

[StT70] R.H. Stolt and J.R. Taylor, Classification of paraparticles, Phys. Rev. **D1** (1970) 2226–2228.

[TIa] H. Tamura and K.R. Ito, A Canonical Ensemble Approach to the Fermion/Boson Random Point Processes and its Applications, to appear in Commun. Math. Phys., available via http://arxiv.org/abs/math-ph/0501053.

[TIb] H. Tamura and K.R. Ito, A Random Point Field related to Bose-Einstein Condensation, available via http://arxiv.org/abs/math-ph/0509071.